Towards understanding the workspace of human limbs

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Significant attention in recent years has been given towards obtaining a better understanding of human joint ranges, measurement, and functionality, especially in conjunction with commands issued by the central nervous system. Studies of those commands often include computer algorithms to describe path trajectories. These are typically in ‘open-form’ with specific descriptions of motions, but not ‘closed form’ mathematical solutions of the full range of possibilities. This paper proposes a rigorous ‘closed form’ kinematic formulation to model human limbs, understand their workspace (also called the reach envelope), and delineate barriers therein where a path becomes difficult or impossible owing to physical constraints. The novel ability to visualize barriers in the workspace emphasizes the power of these closed form equations. Moreover, this formulation takes into account joint limits in terms of ranges of motion and identifies barriers therein where a person is required to attain a different posture. Examples include the workspaces of a typical forearm and a typical finger. The wrist’s range of motion is used to illustrate the visualization of the progress in the functionality of a wrist undergoing rehabilitation.

1. Introduction

The ability to define specific work limitations, physical impairment, changes in limitations with injury or disease, and improvements with therapy have always been problematic, in part owing to their subjective nature. Various authors have proposed more objective measures including specific limitations of motions (Cruse and Bruwer 1987, Uno et al. 1989, Bruwer and Cruse 1990, Kawato et al. 1990, Kawato 1999, Van Thiel et al. 1998) on which impairment is often largely based (e.g., AMA Guides to the Evaluation of Permanent Impairment (AMA 1997)). The reader is also referred to a recent comparison of commercially available measuring systems (Richards 1998). However, in a given patient, none of these approaches can identify the entire range of limitations given specific constraints in particular directions. Consider the tracking of the point of a finger in space. The volume generated by every possible point touched by this finger is called the workspace of that limb. The

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boundary to the workspace is typically called the ‘reach envelope’ (Li and Xi 1990, Molenbroek 1998). Complete identification of workspace is important to:

1. Understand neural strategies allowing the positioning and orienting of the hand during voluntary reaching movements, especially that of the human upper extremity.
2. Quantify the full functional potential of a joint.
3. Study ergonomic postures and motion path trajectories.

The process of moving the hand to a target in space involves a series of sensorimotor transformations that convert the sensory signal of visual data about the location and orientation of the target object (and the arm) into a set of motor commands that will bring the hand to the desired position. The central nervous system (CNS) learns and maintains internal models of these sensorimotor transformations, such as plane or curved trajectories (Breteler et al. 1998) and horizontal plane trajectories (Suzuki et al. 1997). While various models simulate this task and predict postures (for example JACK (Badler 1997)), most of these models rely upon experimental data or incomplete (i.e., ‘open-form’ numerical simulations. These simulations describe specific situations, but owing to lack of formal description (i.e., ‘closed form’ simulations) cannot simulate the entire range of possibilities nor can they address issues pertaining to the ability to cross barriers within the reach envelope.

Impaired arms exhibit well-defined workspace deficits (Reinkensmeyer et al. 1999). This suggests range of motion and constraint forces may prove useful for precise monitoring of arm impairment and the effects of treatment techniques targeted at abnormal synergies and workspace deficits (Kirstukas et al. 1992, Johnson et al. 1999, Yu and Donaldson 1999). The results of the first study were consistent with the standard flexion and extension ‘synergies’ described in the clinical literature. Indeed, this type of approach has been used for the evaluation of arm impairment after brain injury (Reinkensmeyer et al. 1999). Studies of limitations of joint rotation on the independence of hand rotation (Schillings et al. 1998, Kamper and Rymer 1999), while contributing to the discussion of limb workspace and ergonomics have been limited by the number of degrees of freedom (DOF) modelled and by the numerical algorithms used to generate the workspace.

A comprehensive human model using the Visible Human Dataset (VHD) for the European project CHARM demonstrates the need for a better understanding of the workspace of human limbs (Kalra et al. 1995, Maurel et al. 1996, Maurel and Thalmann 1999, 2000). Although this research group has addressed the topological aspects of limbs and joints, the generation of the workspace has not been demonstrated.

The understanding of trajectory formations inside the workspace of human limbs is, to a great extent, dependent upon the identification of control barriers that exist as impediments to motion and that may hinder the execution of a planned trajectory. A rigorous mathematical formulation based on kinematics will first be introduced. Because of this formulation, it can be shown that barriers inside the workspace are identified. More importantly, closed form equations of the workspace are established. As a result, a method for quantifying the mobility (functionality) of a joint can be demonstrated for the wrist. Furthermore, it is shown that visualization of the internal structure of the workspace (where various surfaces appear inside the workspace envelope) provides a powerful tool for grasping the limitations.
2. Modelling and formulation

Whereas the anatomy of limbs and their joints is indeed very complex (as evidenced by the debate in the literature on the correct method for modelling joint motion), a combination of single degree-of-freedom joints such as a revolute joint can be employed (e.g., an elbow joint can be represented as a revolute joint, while the wrist joint can be represented by three intersecting revolute joints). For example, if the resultant motion is rotational, the joint will be modelled as a revolute joint. The effect of a spherical joint is modelled as three revolute joints whose axes intersect at the centre of the sphere. Indeed, all anatomical joints can be modelled using basic kinematic pairs. For example, the elbow joint is modelled as a one degree of freedom (DOF) revolute joint, where $q_1$ is the joint variable (figure 1).

In order to obtain a systematic representation of any serial kinematic chain, we define $\mathbf{q} = [q_1 \ldots q_n]^T \in \mathbb{R}^n$ is defined as the vector of $n$-generalized coordinates defining the motion of a limb with respect to another, where $q_i$ is the individual DOF variable. The position vector function (shown in figure 2) generated by a point of interest written as a multiplication of rotation matrices and position vectors is expressed by

$$\mathbf{x}(\mathbf{q}) = \left[ \begin{array}{c} x(\mathbf{q}) \\ y(\mathbf{q}) \\ z(\mathbf{q}) \end{array} \right] = \sum_{i=1}^{i=n} \left[ \Pi_{j=1}^{j=i-1} j^{-1}\mathbf{R}_j \right]^{i-1} \mathbf{p}_i$$

(1)

where both $i^{-1}\mathbf{p}_i$ and $j^{-1}\mathbf{R}_j$ are defined using the Denavit-Hartenberg (D-H) representation method (Denavit and Hartenberg 1955, Paul 1981, Fu et al. 1987) such that

Figure 1. Definition of a joint and ranges of motion with an elbow modelled as a revolute joint as an example.
\[
\begin{bmatrix}
\cos q_i & -\cos z_i \sin q_i & \sin z_i \sin q_i \\
\sin q_i & \cos z_i \cos q_i & -\sin z_i \cos q_i \\
0 & \sin z_i & \cos z_i
\end{bmatrix}
\]
and
\[(^{(i-1)}R_i = [a_i \cos q_i \ a_i \sin q_i \ d_i]^T\]

where \(q_i\) is the joint angle from \(x_{i-1}\) axis to the \(x_i\) axis, \(d_i\) is the shortest distance between \(x_{i-1}\) and \(x_i\) axes, \(a_i\) is the offset distance between \(z_i\) and \(z_{i-1}\) axes, and \(a_i\) is the offset angle from \(z_{i-1}\) and \(z_i\) axes.

The vector function \(x(q)\) characterizes the set of all points touched by the point of interest. The aim is to determine the envelope of this set. At a specified position in space given by \((x_p, y_p, z_p)\), Equation 1 can be written as a constraint function as

\[
\begin{bmatrix}
x(q) - x_p \\
y(q) - y_p \\
z(q) - z_p
\end{bmatrix} = 0
\]

In mathematical terms, the expression defined by Equation (3) is indeed a Manifold with boundary and cannot readily be visualized.

Joint limits (ranges of motion) are imposed in terms of inequality constraints in the form of

\[q_i^L \leq q_i \leq q_i^U\]

where \(q_i^L\) is the lower limit and \(q_i^U\) is the upper limit, and where \(i = 1, \ldots, n\), with \(n\) as the number of DOFs. As shown in figure 1, a kinematic chain in home configuration is characterized by a zero position as \(q_i = 0\). Using the right hand rule of this joint axis, the joint in counter clockwise is defined as positive (Denavit and Hartenberg 1955), as indicated by \(q_i^L\) and \(q_i^U\). Note that one joint could have more than one degree of freedom (e.g., wrist joint is modelled as three DOFs). The total angles of joint motion are from the measurement results of Norkin and White (1995). In order to include these joint limits in the formulation, the inequalities above are
transformed into equalities by introducing a new set of generalized coordinates \( \lambda = [\lambda_1 \ldots \lambda_n]^T \) such that

\[
q_i = ((q_i^U + q_i^L)/2) + ((q_i^U - q_i^L)/2) \sin \lambda_i \quad i = 1, \ldots, n
\]

where if \( \sin \lambda_i = 1 \), then \( q_i = q_i^U \) and when \( \sin \lambda_i = -1 \), then \( q_i = q_i^L \). In order to include the effect of joint limits, it is proposed to augment the constraint equation with the parameterized inequality constraints of Equation (5) such that

\[
H(q^*) = \left[ \begin{array}{c}
x(q) - x_p \\
y(q) - y_p \\
z(q) - z_p \\
q_i - (q_i^U + q_i^L)/2 - (q_i^U - q_i^L)/2 \sin \lambda_i
\end{array} \right] = 0 \quad i = 1, \ldots n
\]

where \( q^* = [q^T \lambda^T]^T \) is the vector of all generalized coordinates. Note that although \( n \) new variables \( \lambda_i \) have been added, \( n \) equations have also been added to the constraint vector function without losing the dimensionality of the problem.

The Jacobian (named after the German Mathematician Carl G. Jacobi) of the constraint function \( H(q^*) \) at a specific point \( q^0 \) is the \((3 + n) \times 2n\) matrix

\[
H_{q^*} = \frac{\partial H}{\partial q^*}
\]

where the subscript denotes a derivative. Note that the Jacobian is defined in mathematical terms as the derivative of the transformation (Taylor and Mann 1972) between \( x \) and \( q \). With the modified formulation including joint limits, the Jacobian is expanded as

\[
H_{q^*} = \left[ \begin{array}{c|c}
x_q & 0 \\
\hline
I & q_{\lambda}
\end{array} \right]
\]

where \( q_{\lambda} = \partial q/\partial \lambda \), \( x_q = \partial x/\partial q \), \( 0 \) is a \((3 \times n)\) zero matrix, \( I \) is the identity matrix, and

\[
x_q = \left[ \begin{array}{cccc}
x_{q_1} & x_{q_2} & \ldots & x_{q_n} \\
y_{q_1} & y_{q_2} & \ldots & y_{q_n} \\
z_{q_1} & z_{q_2} & \ldots & z_{q_n}
\end{array} \right]
\]

\[
q_{\lambda} = \left[ \begin{array}{cccc}
-((q_1^U - q_1^L)/2) \cos \lambda_1 & 0 & \ldots & 0 \\
0 & -((q_2^U - q_2^L)/2) \cos \lambda_2 & \ldots & 0 \\
0 & 0 & \ldots & -((q_n^U - q_n^L)/2) \cos \lambda_n
\end{array} \right]
\]

Because the Jacobian is not square (more than three DOFs), rank deficiency criteria were developed for surfaces that are swept in space (Abdel-Malek and Yeh 1997) and for robotic arms (Abdel-Malek et al. 1997), and these will be used to obtain all singular behaviour of the Jacobian. Before addressing these criteria, however, it is important to show why the singularity of the Jacobian has a direct effect on identifying the reach envelope. The Jacobian matrix is defined as the transformation matrix that relates (or maps) joint variables (also called joint space) to Cartesian
space. This means that one can now write the joint rates of change of an upper extremity as radians/s and, using the Jacobian matrix, is now able to immediately calculate the hand velocity in terms of the world coordinate system as m/s. The Jacobian matrix is undoubtedly the most important transformation relating the kinematics of the articulated human body. A singularity of this Jacobian matrix means that it has no inverse in the mathematical sense, and this means that, if given the velocity of the hand in Cartesian space and if the Jacobian is singular, it is not possible to calculate the joint velocities, a case that is called singular, which for computer simulation gives rise to several control problems.

This same concept can be used to identify boundary surfaces called singular surfaces that are due to the Jacobian singularities. It will be shown that these singular surfaces are similar in nature to those generated by joint ranges of motion, when a joint has reached its physical limit, yet are different in the sense that Jacobian singularities yield joint angles associated with parallel segmental links of the human body; for example, an arm fully extended characterizes a singularity at the elbow.

A singularity (in the pure mathematical sense) is when the Jacobian has no inverse, i.e., a solution cannot be found. To further explain, consider the differentiation of Equation (1) with respect to time as

$$\dot{x} = x_q \dot{q}$$

where $\dot{q}$ is the vector of joint velocities. Given the hand velocity (i.e., given $\dot{x}$), the calculation of $\dot{q}$ requires computing an inverse of the Jacobian $x_q$. For a singular Jacobian, it is not possible to compute the required velocities for such a path. It will be observed that such behaviour is associated with barriers within the reach envelope (e.g., when the arm is fully extended and cannot extend any further, or when some joints in the arm have reached their limits).

The idea of a singular Jacobian will be used to identify all barriers inside and on the boundary of the workspace. Because the Jacobian is non-square, such barriers are defined as a subset of the workspace at which the Jacobian of the constraint function of Equation (7) is row rank deficient; i.e., barriers defined by $\partial W$ and characterized by

$$\partial W \subset \{ \text{Rank } H_q(q^*) < k, \text{ for some } q^* \text{ with } H(q^*) = 0 \}$$

where $k$ is at least $(3 + n)$. Because of the form of the Jacobian characterized by Equation (12), three distinct conditions arise:

1. **Type I singularity sets:** If no joints have reached their limits, the diagonal sub-matrix $q_{l,q}$ is a full row rank. Therefore, the only possibility for $H_q(q^*)$ to be row-rank deficient is when the block matrix $x_q$ is row rank deficient. The type I singularity set is defined as

$$S^{(1)} \equiv \{ p \in q : \text{Rank}[x_q] < 3, \text{ for some constant subset of } q^* \}$$

where $p$ is within the specified joint limit constraints and may contain joints that are functions of others or constant values.

2. **Type II singularity sets:** When certain joints reach their limits, e.g., $\partial q^\text{lim} = [q_{l,\text{limit}}, q_{j,\text{limit}}, q_{k,\text{limit}}]^T$, the corresponding diagonal elements in the
matrix $\mathbf{q}_2$ will be equal to zero. Therefore, the corresponding matrix is subjected to the rank-deficiency criterion, where $H_{q^*}$ will take on the following form

$$
H_{q^*} \sim \begin{bmatrix}
  x_{q_1} & \cdots & x_{q_i} & x_{q_i} & \cdots & x_{q_k} & \cdots & x_{q_n} \\
  0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
  0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
  0 & \cdots & 0 & 0 & 1 & \cdots & 0
\end{bmatrix}
$$

(14)

and where the three columns pertaining to $x_{q_i}, x_{q_j},$ and $x_{q_k}$ are removed so that the rank deficiency criteria are applied again. From the foregoing observation, the second type of singular sets are formulated. A new vector $\partial \mathbf{q}_{\text{limit}} = [q_i^{\text{limit}}, q_j^{\text{limit}}, q_k^{\text{limit}}]^T$ is defined which is a sub-vector of $\mathbf{q}$ where $1 \leq \dim(\partial \mathbf{q}_{\text{limit}}) \leq (n-3)$. The type II singularity set is defined as

$$
S^{(2)} \equiv \left\{ \mathbf{p} = [\mathbf{p} \cup \partial \mathbf{q}_{\text{limit}}] ; \text{Rank}[\mathbf{x}_\mathbf{q}(\mathbf{w}, \partial \mathbf{q}_{\text{limit}})] < 3, \text{for some } \mathbf{p} \in \mathbf{q}^*, \dim(\partial \mathbf{q}_{\text{limit}}) \leq (n-3) \right\}
$$

(15)

where $\mathbf{p}$ is the singular set as a result of applying the rank deficiency criteria to Equation (14).

3) Type III singularity sets: These are all sets that are composed of the combination of joints at their limits and are defined by:

$$
S^{(3)} \equiv \left\{ \mathbf{p} \in \mathbb{R}^{(n-2)}; \mathbf{p} \equiv \partial \mathbf{q}_{\text{limit}} = [q_i^{\text{limit}}, q_j^{\text{limit}}, \cdots] ; \text{where } i \neq j \right\}
$$

(16)

Barriers are identified by substituting the sets $\mathbf{p}_i$ characterized by Equations (13, 15, and 16) into the accessible set $\mathbf{x}(\mathbf{q})$ which yields the equation of a surface that can be readily shown. This surface is indeed a barrier associated with a generalized variable that has reached its limit. Determining joint angles of a limb given a specific position and orientation is usually defined as the inverse kinematics problem in the robotics literature (Fu et al. 1987). Motion from one configuration to another along a trajectory sometimes requires halting the motion and changing the inverse kinematics in order to proceed with the motion. An example of this occurs when attempting to reach a point located behind one’s shoulder. Starting with one trajectory may become very uncomfortable because of joint limits, while trying another trajectory becomes simpler. Similarly, reaching a doorknob and turning is sometimes difficult to complete and requires orienting the initial hand configuration in a different posture. These barriers due to singular sets identified by Equations (13), (15) and (16) may admit motion only in one normal direction, and hence are called impediments to motion (Abdel-Malek et al. 1999). In this case, the arm, for example, will not be able to cross such a barrier.

3. Illustrative example

Before addressing a model of the upper extremity with many DOFs, an illustration will be given of the formulation of an arm that is limited to planar motion (e.g., on the surface of a table). Consider the motion of the shoulder and the elbow where both joints are parallel (both revolute joints) and their axes perpendicular to the surface of the table as shown in figure 3. The point of interest is on the tip of the index finger.
Coordinates of the point of interest \( P \) located at the tip of the index finger can be written as

\[
x(q) = \begin{bmatrix} -10 \sin q_1 - 13 \sin (q_1 + q_2) \\ 10 \cos q_1 + 13 \cos (q_1 + q_2) \end{bmatrix}
\]  
with joint limits imposed as \(-120^\circ \leq q_1 \leq 75^\circ\) and \(-5^\circ \leq q_2 \leq 150^\circ\), where the tip of the index finger is positioned at \((5, 0)\) with respect to the fourth coordinate system. These joint ranges of motion are converted into equality constraints as:

\[
q(\lambda) = \begin{bmatrix} (75 - 120)^\circ \pi/(2^\circ 180) + (75 + 120)^\circ \pi/(2^\circ 180) \sin \lambda_1 \\ 1.265 + 1.353 \sin \lambda_2 \end{bmatrix}
\]

In order to compute the Jacobian singular behaviour, we first calculate the Jacobian

\[
x_q = \begin{bmatrix} -10 \cos q_1 - 13 \cos (q_1 + q_2) & -13 \cos (q_1 + q_2) \\ -10 \sin q_1 - 13 \sin (q_1 + q_2) & -13 \sin (q_1 + q_2) \end{bmatrix}
\]

and

\[
q_q = \begin{bmatrix} -1.700 \cos \lambda_1 & 0 \\ 0 & -1.353 \cos \lambda_2 \end{bmatrix}
\]

The singularity of the Jacobian is computed by setting the determinant of \( x_q \) to zero (since it is a square matrix, otherwise the determinant of all square sub-Jacobians will be set to zero). The determinant of the Jacobian is

\[
|x_q| = 130 \sin q_2
\]

Setting Equation (21) to zero yields a solution as \( q_2 = 0^\circ \) (note that while \( q_2 = 180^\circ \) is also a solution, it does not satisfy the joint range constraints). Similarly,
applying the same criteria to $q_2$ yields the additional singularities (which are indeed the joint limits in this case). Substituting each joint limit into Equation (17) yields a curve. For example, for $q_2 = 0$, the resulting equation is given by $x^{(1)}(q_1, q_2 = 0) = \frac{-23}{23} \sin q_1 \cos q_1$; (at $q_1 = q_L^r$ for $120^\circ \leq q_1 \leq 75^\circ$ and shown in figure 4.

Similarly, substituting each singularity into $x(q)$ yields a curve. The reach envelope is shown by plotting all curves in figure 5.

4. Workspace of the upper extremity
Consider the shoulder and forearm modelled as a four-DOF system, where the spherical joint at the shoulder is modelled as three intersecting revolute joints and the elbow as a revolute joint. This is consistent with published results except that the wrist joint (which is an additional three revolute joints) has not been considered and we have limited the motion of the glenohumeral joint to spherical. Furthermore, the spherical joint has been modelled as three revolute joints intersecting at one point, a practice commonly used in modelling to represent

![Figure 4. Singular curve due to $q_2 = 0$.](image)

![Figure 5. The workspace envelope of the planar arm.](image)
spherical joints. It should be noted that the most difficult and the least successful modelling of a major articulating joint has been the shoulder because of the lack of appropriate biomechanical data as well as the anatomical complexity of the region. Figure 6 depicts the joint motions to be modelled where each joint is given an independent coordinate $q_i$ where the equivalent kinematic skeleton of the system is depicted with the $z$-axis located according to the Denauit-Hartenberg representation method ($a = b = 10\degree$, $c = 5\degree$ for a 95% male). The dimensions of the arm are also noted on the figure.

In the following analysis, a point on the tip of the thumb as shown in figure 7 will be tracked. In the field of kinematics, the motion of a spherical joint with three DOFs can be modelled as three independent revolute joints having their axes intersecting at a single point as shown in figure 7. Note that the point on the thumb is shown located at the position $^4v = [5 0 5]^T$ as resolved in the fourth coordinate frame.

It should also be noted that this model is limited only to the glenohumeral and not the scapulothoracic motion of the shoulder joint (i.e., the additional three translational DOFs of the scapulothoracic joint are not taken into consideration).

In order to demonstrate the formulation, consider the following joint limits imposed on the model of figure 6: $-90^\circ \leq q_1 \leq 90^\circ$, $-110^\circ \leq q_2 \leq 120^\circ$, $-90^\circ \leq q_3 \leq 90^\circ$, and $-150^\circ \leq q_4 \leq 0^\circ$. Using the Denavit-Hartenberg representation method, the thumb position is given by Equation (1) as

![Figure 6. Shoulder and arm and the corresponding degree of freedom.](image_url)
Rank deficiency criteria applied to the resulting \((3 \times 4)\) Jacobian matrix yields 40 singular sets that are listed in the Appendix. Note that most singular sets contain variables that use functions of other variables, i.e., coupled behaviour. For example, substituting singular set \(p_1: (q_1 = -\pi/2\) and \(q_2 = -110^\circ)\) into Equation (17) yields an exact closed form equation of a barrier as

\[
x(q) = \begin{bmatrix}
5\{\cos q_3\sin q_1 - \cos q_1(4 + 5\cos q_4)\sin q_2\} + (4 + 5\cos q_4)\sin q_1\sin q_3\cos q_4\sin q_1\sin q_3 - 5\cos q_2\sin q_4) \\
-5\{\cos q_1\cos q_3 + (4 + 5\cos q_4)\sin q_4\sin q_1\cos q_3\sin q_3 - (4 + 5\cos q_4)\sin q_2\sin q_3 + 5\cos q_2\sin q_4) \\
5\{\cos q_2\cos q_3(4 + 5\cos q_4) - \sin q_3\} - 5\sin q_2\sin q_4)
\end{bmatrix}
\]

In order to explain the physical meaning of singular surfaces (barriers to motion), the skeleton model shown in figure 8 at the given singular configuration \((q_1 = 1–90^\circ\) and \(q_2 = -110^\circ)\) can be used, so that only \(q_3\) and \(q_4\) are allowed to vary. The surface shown in figure 7 is a geometric entity in space where the thumb is permitted to move. These surfaces may exist inside and on the boundary of the workspace. Some of these surfaces present impediments to motion because the arm will not admit motion in one of the normal directions (Abdel-Malek et al. 1999). Combining all 40 barriers yields the workspace of the forearm as shown in figure 9 (where two cross-sectional views are shown).

The workspace oriented with respect to the torso is shown in figure 10.
4.1. Crossing a barrier

Some of the singular surfaces identified in figure 10 are impediments to motion. It was shown in recent work (Abdel-Malek et al. 1999) that when a configuration that does not admit motion in the direction normal to the surface (only in one direction), the surface is called a barrier. To explain a barrier, consider the configuration of the arm where the hand is positioned behind the shoulder (as shown in figure 11). While this point is still in the workspace, no further motion is allowed. However, this same point can be reached with a different configuration of the arm, and yet the motion of
the arm can continue beyond that point. Therefore, barriers are imaginary surfaces where a trajectory followed by the arm is interrupted because of the inability to find an admissible inverse kinematic solution through the surface. Figure 11 shows two views of the human, where the arm is at a barrier that it cannot cross, yet is in the reach envelope, but requires a different posture.

5. **Finger workspace**

Consider the workspace of a point located at the tip of the index finger as shown in figure 12a. The kinematic motion of the finger is modelled as four revolute joints,
two of which intersect as shown in figure 12b. Limits for the finger joints are as follows: $0^\circ \leq q_1 \leq 20^\circ$, $-30^\circ \leq q_2 \leq 50^\circ$, $0^\circ \leq q_3 \leq 90^\circ$, and $0^\circ \leq q_4 \leq 70^\circ$. The complete workspace (every point touched by $P$) is shown in figure 13.

6. Wrist’s range of motion (quantifying the workspace)

As a direct application of this formulation, it is now possible to visualize the progress of a certain joint via workspace analysis. The measurement techniques and devices are well established. However, the range of motion is typically given in terms of a set of numerical joint angle values. The progress is difficult to monitor. The following example will show that it is now possible to visualize the progress through a series of plots that depict the mobility of the joint (its workspace). Consider for example the wrist joint and hand shown in figure 14, where the wrist has been modelled as a three-DOF system.

For an individual who has had a surgical procedure, the wrist joint motion may take weeks or months to return to normal or may be left with residual restrictions. Progress made, whether due to time alone or physical therapy, is measured by the ranges of motion. Using the above formulation, not only visualization of the progress can effectively be made, but an accurate overall number can be used. Indeed, the surface area (or volume if the workspace is a volume), can be used to provide a good estimate. A normal joint range of motion for an adult is $-180^\circ \leq q_1 \leq 45^\circ$, $-70^\circ \leq q_2 \leq 80^\circ$, and $-20^\circ \leq q_3 \leq 40^\circ$, where the initial configuration of the hand is given as horizontal, with the thumb up, and the arm extended and away from the body. Using the above formulation, the resulting workspace is indeed a surface (a region of a spherical surface) as shown in figure 15.

For a person who is undergoing physical therapy after a surgical operation, the functionality of the wrist may first be very limited. For example, immediately after the operation, the wrist joints may be limited to $-90^\circ \leq q_1 \leq 10^\circ$, $-30^\circ \leq q_2 \leq 30^\circ$, and $-10^\circ \leq q_3 \leq 20^\circ$, for which the workspace is shown in figure 16a. As the joint gains better mobility, the range of motion is increased and the progress is monitored by the visual workspace as shown in figure 16 (b-d). If an accurate measure is needed, the surface area is obtained by an integration over the surface. However this is only possible because of the ability to obtain equations of the boundary.
7. Discussion

A better understanding of the workspace of human limbs will aid researchers in improving the understanding of the central nervous system and the manipulation of motor commands. It was shown that the reach envelope of an approximate articulated human model is determined in closed form (i.e., equation of surfaces on the envelope are determined), in addition to the so-

Figure 13. Workspace of the finger.

Figure 14. (a) A schematic diagram of the wrist and hand (b) Modelling of the wrist joint.
Figure 15. The workspace of a point on the tip of the thumb with respect to the wrist.

Figure 16. Various workspace surfaces of the wrist at different joint limits (a) \(-90^\circ \leq q_1 \leq 10^\circ, -30^\circ \leq q_2 \leq 30^\circ,\) and \(-10^\circ \leq q_3 \leq 20^\circ,\) (b) \(-120^\circ \leq q_1 \leq 20^\circ,\) \(-40^\circ \leq q_2 \leq 45^\circ,\) and \(-13^\circ \leq q_3 \leq 25^\circ\) (c) \(-140^\circ \leq q_1 \leq 30^\circ, -50^\circ \leq q_2 \leq 60^\circ,\) and \(-16^\circ \leq q_3 \leq 32^\circ,\) (d) \(-160^\circ \leq q_1 \leq 40^\circ, -60^\circ \leq q_2 \leq 70^\circ,\) and \(-18^\circ \leq q_3 \leq 37^\circ\).
called singular surfaces within the envelope. These surfaces are where the kinematic motion is impeded because of joint locks (known as gimbal locks in mechanics, or singular configurations in robotics). These surfaces are indeed barriers to motion, where in some cases, crossing or traversing these barriers with a specific configuration is not possible (see the crossability analysis for robotics, Abdel-Malek et al. 2001). Similar work for crossability based on the surfaces identified for humans in this work is well underway and will be reported elsewhere.

The present study has focused on geometric features of workspace and joint-space paths of three-dimensional reaching movements. It has been shown that a rigorous formulation for modelling, analysis, and visualization of the workspace of human limbs can be obtained. It has also been shown that barriers to trajectory control can be explicitly identified in closed form and are based on the singularity of the Jacobian matrix governing the motion. Ranges of motion (as joint limits) were incorporated into the formulation and played an important role in distinguishing these barriers. While this is a first step towards investigating the workspace, only positioning (e.g., reachability of the arm) is considered. Subsequent work will concentrate on a better understanding of the orientation workspace (e.g., wrist dexterity). Workspaces of the forearm and the finger were demonstrated. Because of this unique ability to generate the workspace boundary in closed form, it was shown that progress towards full mobility of a joint undergoing physical therapy, for example, is easily visualized and accurately quantified using a surface area calculation. The potential of this work to play an important role in the design for ergonomics and in providing a quantifiable measure for the functionality of joints is evident.

References

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Appendix

Singularity sets of the shoulder-forearm example:

\[ p_1 = \{q_1 = -\pi/2, q_2 = -11\pi/18\}, \quad p_2 = \{q_1 = -\pi/2, q_2 = 2\pi/3\}, \]
\[ p_3 = \{q_1 = \pi/2, q_2 = -11\pi/18\}, \quad p_4 = \{q_1 = \pi/2, q_2 = 2\pi/3\}, \]
\[ p_5 = \{q_1 = -\pi/2, q_3 = -\pi/2\}, \quad p_6 = \{q_1 = -\pi/2, q_3 = \pi/2\}, \]
\[ p_7 = \{q_1 = -\pi/2, q_3 = -\pi/2\}, \quad p_8 = \{q_1 = -\pi/2, q_3 = \pi/2\}, \]
\[ p_9 = \{q_2 = -11\pi/18, q_3 = \pi/2\}, \quad p_{10} = \{q_2 = -11\pi/18, q_3 = -\pi/2\}, \]
\[ p_{11} = \{q_2 = 12\pi/18, q_3 = -\pi/2\}, \quad p_{12} = \{q_2 = 12\pi/18, q_3 = \pi/2\} \]

The reduced rank deficiency singularity sets are as listed below.

\[ p_{13} = \left\{ q_1 = -\pi/2, q_3 = -\text{Arc cos} \frac{1}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{14} = \left\{ q_1 = \pi/2, q_3 = -\text{Arc cos} \frac{1}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{15} = \left\{ q_2 = -11\pi/18, q_3 = -\text{Arc cos} \frac{1}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{16} = \left\{ q_2 = 2\pi/3, q_3 = -\text{Arc cos} \frac{1}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{17} = \left\{ q_1 = -\pi/2, q_3 = \text{Arc cos} \frac{1}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{18} = \left\{ q_1 = \pi/2, q_3 = \text{Arc cos} \frac{1}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{19} = \left\{ q_2 = -11\pi/18, q_3 = \text{Arc cos} \frac{1}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{20} = \left\{ q_2 = 2\pi/3, q_3 = \text{Arc cos} \frac{1}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{21} = \left\{ q_1 = -\pi/2, q_3 = -\text{Arc cos} \frac{\sqrt{16 + 40 \cos q_4 + 25 \cos^2 q_4}}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{22} = \left\{ q_1 = \pi/2, q_3 = -\text{Arc cos} \frac{\sqrt{16 + 40 \cos q_4 + 25 \cos^2 q_4}}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{23} = \left\{ q_2 = -11\pi/18, q_3 = -\text{Arc cos} \frac{\sqrt{16 + 40 \cos q_4 + 25 \cos^2 q_4}}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{24} = \left\{ q_2 = 2\pi/3, q_3 = -\text{Arc cos} \frac{\sqrt{16 + 40 \cos q_4 + 25 \cos^2 q_4}}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{25} = \left\{ q_1 = -\pi/2, q_3 = -\text{Arc cos} \frac{\sqrt{16 + 40 \cos q_4 + 25 \cos^2 q_4}}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}, \]
\[ p_{26} = \left\{ q_1 = \pi/2, q_3 = -\text{Arc cos} \frac{\sqrt{16 + 40 \cos q_4 + 25 \cos^2 q_4}}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \right\}. \]
\[ p_{27} = \begin{cases} q_2 = -11\pi/18, & q_3 = \arccos \frac{\sqrt{16 + 40 \cos q_4 + 25 \cos^2 q_4}}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \\ p_{28} = \begin{cases} q_2 = 2\pi/3, & q_3 = \arccos \frac{\sqrt{16 + 40 \cos q_4 + 25 \cos^2 q_4}}{\sqrt{17 + 40 \cos q_4 + 25 \cos^2 q_4}} \\ p_{29} = \begin{cases} q_1 = -\pi/2, & q_2 = -\arcsin \frac{5 \sin q_4}{\sqrt{1 + 25 \sin^2 q_4}} \\ p_{30} = \begin{cases} q_1 = -\pi/2, & q_2 = -\arcsin \frac{5 \sin q_4}{\sqrt{1 + 25 \sin^2 q_4}} \\ p_{31} = \begin{cases} q_3 = -\pi/2, & q_2 = -\arcsin \frac{5 \sin q_4}{\sqrt{1 + 25 \sin^2 q_4}} \\ p_{32} = \begin{cases} q_1 = \pi/2, & q_2 = \arcsin \frac{5 \sin q_4}{\sqrt{1 + 25 \sin^2 q_4}} \\ p_{33} = \begin{cases} q_1 = -\pi/2, & q_2 = -\arcsin \frac{5 \sin q_4}{\sqrt{1 + 25 \sin^2 q_4}} \\ p_{34} = \begin{cases} q_1 = \pi/2, & q_2 = \arcsin \frac{5 \sin q_4}{\sqrt{1 + 25 \sin^2 q_4}} \\ p_{35} = \begin{cases} q_3 = \pi/2, & q_2 = \arcsin \frac{5 \sin q_4}{\sqrt{1 + 25 \sin^2 q_4}} \\ p_{36} = \begin{cases} q_1 = -\pi/2, & q_4 = \arcsin \frac{\tan q_2}{5} \\ p_{37} = \begin{cases} q_1 = \pi/2, & q_4 = \arcsin \frac{\tan q_2}{5} \\ p_{38} = \begin{cases} q_3 = -\pi/2, & q_4 = \arcsin \frac{\tan q_2}{5} \\ p_{39} = \begin{cases} q_3 = \pi/2, & q_4 = \arcsin \frac{\tan q_2}{5} \\ \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \]